SOME CONSIDERATIONS ON THE NINE-POINT CONIC AND ITS RECIPROCAL.

By Miss Fanny Gates, Waterloo, Iowa.

In Volume VII of this Journal, Mr. Holgate enunciated the theorem on the nine-point conic in its most general form as follows :—

"Let $ABCD$ be any complete quadrangle whose six sides AB, AC, AD, BC, BD, CD are cut by an arbitrary straight line a, in the points P, Q, R, S, T, V, respectively, and let E, F, H, K, L, M, be the harmonic conjugates of these points with respect to the pairs of vertices of the quadrangle, so that $AEBP$, $AFCQ$, etc., are harmonic ranges. Then a conic may be passed through the six points E, F, H, K, L, M, on which will also lie the three points of intersection X, Y, Z, of the pairs of opposite sides of the quadrangle."

It is well known that the locus of the poles of an arbitrary straight line a with respect to the system of conics through four points, is a conic, (See Smith's Conics, p. 236, Ex. 2), but it does not appear to have been observed that this locus is the nine-point conic determined, as above, by this straight line. This may be readily proved in the following manner :—

Choose AB and CD as the axes of coordinates, and let the equations of AD and BC be

$$ax + by - 1 = 0 \quad \text{and} \quad a'x + b'y - 1 = 0, \quad \text{respectively,}$$

while the equation of the arbitrary straight line a is

$$lx + my - 1 = 0 .$$

The equation of the system of conics through the four points A, B, C, D, will be

$$(ax + by - 1)\,(a'x + b'y - 1) - \lambda xy = 0 .$$

Identify the line a with the polar of an arbitrary point (x', y') with respect to this system of conics, and we obtain as the locus of this point, the conic

$$(2aa' - al - a'l)\,x^2 + (am + a'm - bl - b'l)\,xy - (2bb' - bm - b'm)\,y^2$$
$$- (a + a' - 2l)\,x + (b + b' - 2m)\,y = 0 .$$

This conic evidently passes through the intersection of AB and CD, and similarly through the intersections of the other pairs of opposite sides of the quadrangle. It cuts the line $x = 0$ a second time at the point where

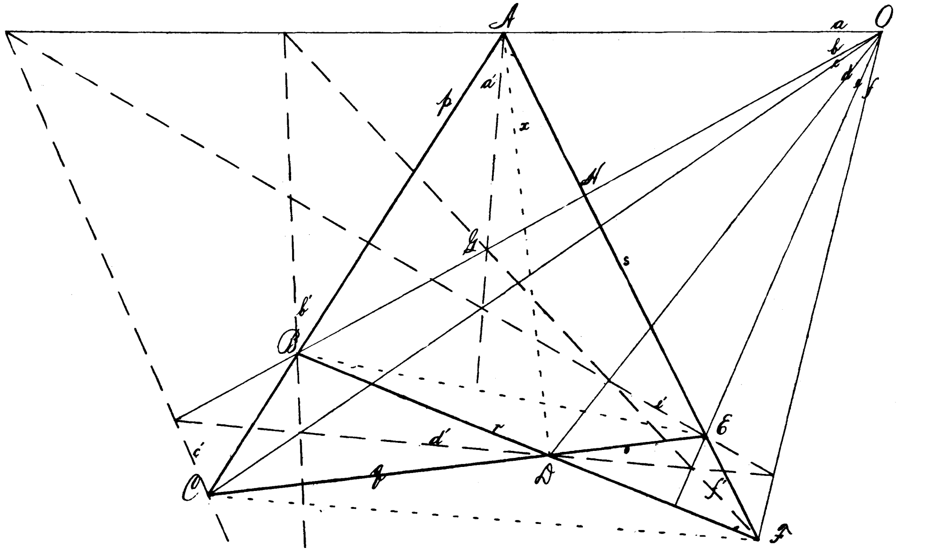
$y = \dfrac{b + b' - 2m}{2bb' - bm - b'm}$. But this point is the harmonic conjugate of the point V (in which the line a cuts the y axis) with respect to the two vertices C and D lying on this side.

In like manner, it may be shown that the locus of the poles of the line a passes through the harmonic conjugate points marked out on the other sides of the quadrangle.

By taking the reciprocal of this general statement of the nine-point conic, we obtain the following interesting theorem which admits a simple geometric proof :—

If the six vertices of a complete quadrilateral be projected from any point O, of the plane, and the harmonic conjugates of these rays with respect to the pairs of sides passing through the same vertex be found, then these six conjugate rays are tangent to one conic, to which also the three diagonals of the quadrilateral are tangent.

Let A, B, C, D, E, F be the six vertices of any complete quadrilateral p, q, r, s, such that A, B, C lie on p, C, D, E on q, B, D, F on r and A, E, F on s.



Project these vertices from any point O of the plane by the rays a, b, c, d, e, f. Let the harmonic conjugates of these rays with respect to the two sides of the quadrilateral passing through the same vertex be a', b', c', d', e', f', respectively.

Then a' and f' will intersect on b. For, if the ray b intersect the side s in H and a' in G, then $OHGB$ is a harmonic range, and if b intersect f' in G' then $OHG'B$ is a harmonic range. Therefore G and G' coincide, that is, a' and f' intersect on b. Likewise b' and f' intersect on a, b' and d' on c, and so also all pairs of similarly situated lines.

Now the rays a', e', c', d', b', f' form a hexagon, such that

a' and e', d' and b' intersect on c,
e' " c', b' " f' " " a,
c' " d', f' " a' " " b;

and since a, b, and c pass through one point O, these lines are tangent to one conic. (Brianchon's Theorem.)

Moreover, if x be any one of the three diagonals of the quadrilateral, say AD, then the rays a', e', f', b', d', x form a hexagon, such that

a' and e', b' and d' intersect on c,
e' " f', d' " x " " d,
f' " b', x " a' " " a;

and since a, c, and d are concurrent, this hexagon circumscribes a conic, that is, the diagonal x, and similarly, each of the other diagonals of the quadrilateral, is tangent to the conic determined by the five rays a', e', f', b', d'. But this is the same conic to which we have shown the ray c' to be tangent. In other words, the six conjugate rays a', b', c', d', e', f' and the three diagonals of the quadrilateral are all tangent to the same conic.

The complete system of tangents to this conic is simply the system of polars of the point O, with respect to the system of conics touching the four sides of the quadrilateral.

This follows directly as the reciprocal of the property proved above, that the locus of the poles of an arbitrary straight line, with respect to the system of conics through four points, is the nine-point conic, determined by the straight line; or it may be readily shown analytically.

The above theorem admits special cases similar to those of the nine-point conic. These arise in accordance with the relative positions of the point O, and the four sides of the quadrilateral.

An analogous configuration in three dimensional space would be the following:—

Let $ABCD$ be any tetrahedron whose edges AB, AC, AD, BC, BD, CD, are projected from an arbitrary point O, not lying in any face of the tetrahedron, by the planes α, β, γ, δ, ε, ζ, respectively, and let the harmonic

conjugates of these planes with respect to the two faces of the tetrahedron passing through the same edge, be α', β', γ', δ', ε', ζ'.

If π be any plane through the point O, it will cut the planes α', β', γ', δ', ε', ζ' in six lines, which are tangent to one conic, to which also are tangent the three lines joining the points where pairs of opposite edges of the tetrahedron are cut by the plane π.

The lines of intersection of the planes α' and ζ', β' and ε', γ' and δ', which pass through the pairs of opposite edges of the tetrahedron, lie in one plane.

The demonstration of this theorem is wholly similar to that enunciated concerning the quadrilateral. In fact, the configuration of the plane theorem is identical with that marked out in the plane of section π.

NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS, *Feb.*, 1894.